Engineering Notes

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Avoidance of Numerical Instabilities in Computation of Higher-Order Gravity

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Introduction

THE calculation of higher-order gravity involves the solution of Laplace's equation in spherical coordinates. The solution requires the evaluation of associated Legendre functions. These functions can be computed explicitly or via recursion relationships. The use of the explicit relations necessitates excessive computer execution time or memory. On the other hand, uncritical use of standard recursion relationships, particularly on limited word-length machines, can lead to numerical instabilities near the poles. Alternative applications of recursion relationships are presented that avoid instabilities while retaining the advantages of minimal computer execution time and memory.

Discussion

The gravitational potential outside of the Earth satisfies Laplace's equation and can be developed as an infinite series in spherical harmonics.[1,2] The infinite series is truncated to finite order and degree in practical applications, as follows:

$$U=\frac{GM}{r}\sum_{n=0}^{N}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n}P_n^m(\sin\phi)\left[C_{nm}\cos m\lambda+S_{nm}\sin m\lambda\right] \tag{1}$$

where r is the radial distance of the external point from the center of mass of the Earth, ϕ the geocentric latitude, λ the longitude, a the semimajor axis of the Earth, M its mass, G the universal gravitational constant, and C_{nm}, S_{nm} are expansion coefficients derived from observation. The gravitational acceleration is simply the gradient of the potential in Eq. (1).

The P_n^m are the associated Legendre functions and can be determined from the following expression:[3]

$$P_n^m(x)=2^{-n}(1-x^2)^{m/2}\sum_{k=0}^{r}(-1)^k \times\frac{(2n-2k)!}{k!(n-k)!(n-m-2k)!}x^{n-m-2k} \tag{2}$$

where r is the greatest integer less than or equal to $(n-m)/2$ and $x=\sin\phi$. Equation (2) can be used directly to compute P_n^m. However, this approach results in excessive computer execution time. Alternatively, Eq. (2) can be solved and the results expressed in terms of polynomials in powers of x. The resulting polynomials can then be stored in the computer. This approach alleviates the execution time problem, but increases computer memory requirements.

Consider now the use of recursion relationships for the generation of P_n^m. Such relationships have the advantages of both execution speed and minimal memory requirements. As an example of how certain formulations can lead to undesirable numerical results, a standard recursion relationship used for generation of associated Legendre functions is

$$P_n^{m+1}=m\alpha P_n^m-(n+m)(n-m+1)P_n^{m-1} \tag{3}$$

where $\alpha=2x/\sqrt{1-x^2}$.

The usual use of this relation is in the forward sense. That is, starting values are given for P_n^0 and P_n^1. Equation (3) is then used to compute P_n^m for increasing m up to $m=n$. It will be shown that this algorithm leads to numerical instabilities in the region near $x^2=1$.

The above assertion can easily be demonstrated with the following argument. Consider the ratio R_f where the numerator and denominator are the terms on the right-hand side of Eq. (3),

$$R_f=\frac{m\alpha P_n^m}{(n+m)(n-m+1)P_n^{m-1}}$$

Using Eq. (3), this can be rewritten as

$$R_f=1+R$$

where

$$R=\frac{P_n^{m+1}}{P_n^{m-1}(n+m)(n-m+1)}$$

However, R can be expressed as

$$R=\frac{(1-x^2)\mathrm{d}^{n+m+1}/\mathrm{d}x^{n+m+1}(x^2-1)^n}{(n+m)(n-m+1)\mathrm{d}^{n+m-1}/\mathrm{d}x^{n+m-1}(x^2-1)^n}$$

from which it follows that $R=0$ and thus $R_f=1$ when $x^2=1$. Therefore, we can expect a numerical instability in Eq. (3) in the region near $x^2=1$ since we are subtracting two numbers of nearly equal value to yield a much smaller number. This has been observed in direct application of Eq. (3) for values of x^2 near 1, where calculated values of P_n^m become increasingly large as m is increased. Two alternative techniques for utilizing recursion relations are provided below. By an argument analogous to that following Eq. (3), the alternate mechanizations below can be shown to avoid numerical instabilities in the region near $x^2=1$.

Alternative 1

Direct application of Eq. (2) yields

$$P_n^n=(2n-1)(1-x^2)^{1/2}P_{n-1}^{n-1} \tag{4}$$

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Using the standard recursion relation,

$$(n-m)P_n^m=(2n-1)P_{n-1}^m-(n+m-1)P_{n-2}^m$$

and the fact that $P_n^m=0$ for $m>n$, a simple relation can be obtained for P_{n+1}^n:

$$P_{n+1}^n=(2n+1)P_n^n \tag{5}$$

Then, Eqs. (4) and (5) can be used to compute any P_n^m given the initial value $P_1^1=(1-x^2)^{1/2}$.

Alternative 2

An alternative mechanization of Eq. (3) is

$$P_n^{m-1}=\frac{m\alpha P_n^m-P_n^{m+1}}{(n+m)(n-m+1)} \tag{6}$$

In this case, we first compute all P_n^n up to $n=N$ using Eq. (4) with the starting value $P_1^1=(1-x^2)^{1/2}$. Equation (6) is then solved starting with P_n^n and proceeding with decreasing m to $m=1$ for given n. Where appropriate, we again make use of the relationship $P_n^m=0$ for $m>n$ in utilizing Eq. (6). In contrast to Eq. (3) applied in the forward sense, Eq. (6) is stable when run backward over m. An analogous instability has been discussed for Bessel functions.[4]

Conclusions

Use of the standard recursion relationships for the associated Legendre functions will, in general, lead to numerical roundoff errors in the computed gravity. These roundoff errors can avalanche catastrophically near the poles unless the recursion relations used are stable. Two examples of stable recursion relations have been discussed and an unstable one analyzed.

Acknowledgments

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The Minimum for Geometric Dilution of Precision in Global Positioning System Navigation

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In a previous Note,[1] the author presented some simple bounds to the global positioning system (GPS) navigation performance index, the geometric dilution of precision (GDOP). It was shown that the GDOP must be greater than $\sqrt{2}$ and that a value as low as $\sqrt{2.5}$ was attained for a completely symmetrical GPS configuration; i.e., the line-of-sight vectors from the user to the four GPS satellites were all separated by the same angle $\cos^{-1}(-1/3)$. It will be shown below that the value of $\sqrt{2.5}$ is indeed the minimum for GDOP.

As shown in Ref. 1,

$$\mathrm{GDOP}=\sqrt{1/\lambda_1+1/\lambda_2+1/\lambda_3+1/\lambda_4} \tag{1}$$

where the λ are eigenvalues of a 4×4 real symmetric nonnegative matrix with a trace equal to 8. The results of Ref. 1 were obtained by examining the 4×4 matrix HH^T, but it was pointed out that this 4×4 matrix may also be

$$H^TH=\begin{bmatrix} aa^T+bb^T+cc^T+dd^T & a+b+c+d \\ (a+b+c+d)^T & 4\end{bmatrix} \tag{2}$$

where

$$H^T=\begin{bmatrix} a & b & c & d \\ 1 & 1 & 1 & 1\end{bmatrix}$$

is the measurement partial derivative matrix, transposed; *a, b, c,* and *d* are line-of-sight unit vectors from a set of four GPS satellites to the user. From the well-known mini-max property of the eigenvalues of symmetric matrices,[2] one has, for the largest eigenvalue λ_4 of H^TH,

$$\lambda_4\geq[0\ 0\ 0\ 1]\,(H^TH)\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=4 \tag{3}$$

Subject to this constraint and the fact that all the λ are nonnegative and have a sum equal to 8, one has

$$\mathrm{GDOP}\geq\mathrm{minimum}\sqrt{[1/\lambda_4+3/[(8-\lambda_4)/3]}=\sqrt{2.5}$$

occurring at $\lambda_4=4$. That 2.5 is a minimum and not a lower bound has already been shown by the construction of the completely symmetrical GPS configuration.

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Optimization of Cruise at Constant Altitude

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Nomenclature

C = cost function
C_{D_0} = drag coefficient at zero lift

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